2.15 Monopoles and Dipoles

The influence function for the slope w' of a beam is generated by a single moment $M = 1$ or **dipole**

$$M = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Delta x = 1 \,, \tag{2.118}$$

a pair of equal-sized but opposite forces $\pm 1/\Delta x$ whose distance Δx shrinks to zero.

A single force alone, in contrast, is a **monopole**. The influence function for a deflection $w(x)$ is the reaction to a monopole.

Influence functions which are generated by monopoles **sum**, they resemble dents or sinks, see Figs. 2.42 and 2.45a. Everything which falls into the sink increases the deflection of the plate.

Dipoles instead generate shear deformations, which are sensitive to imbalances, they **differentiate**, see Figs. 2.42 and 2.45b.

Monopoles integrate and dipoles differentiate.

Each of the four influence functions in Fig. 2.42 is of either type:

- G.F. for deflections and moments *sum*.
- G.F. for rotations, stresses and shear forces *differentiate*

The influence function for the shear force V is generated by a dipole, while the influence function for the bending moment M is generated by two moments $M = \pm 1/\Delta x$, which rotate the beam inwards, and so generate a symmetric deflection with a kink at the source point.[1]

The maximum result is obtained if the load and the influence function are of the same type (***symmetric—symmetric*** or ***antisymmetric—antisymmetric***) and the minimum effect when they are of opposite type, see Fig. 2.43.

The difference between monopoles and dipoles is the reason, why it is easier to approximate displacements and bending moments than stresses and shear forces. It is the difference between **numerical integration** and **numerical differentiation**, see Fig. 2.44.

Remark 2.6 Influence functions for support reactions integrate, though the support reactions are normal forces (stresses) or shear fores, and we therefore would expect the influence functions to differentiate. But if the support sits on a rigid soil, the

[1]To be precise: the correct sequence is: monopole—dipole—quadrupole—octopole, corresponding to the finite differences of w, w', M, V, see Fig. 9.11 page 446, but for our purposes the simple division: monopole—dipole or sum—differentiate will do.

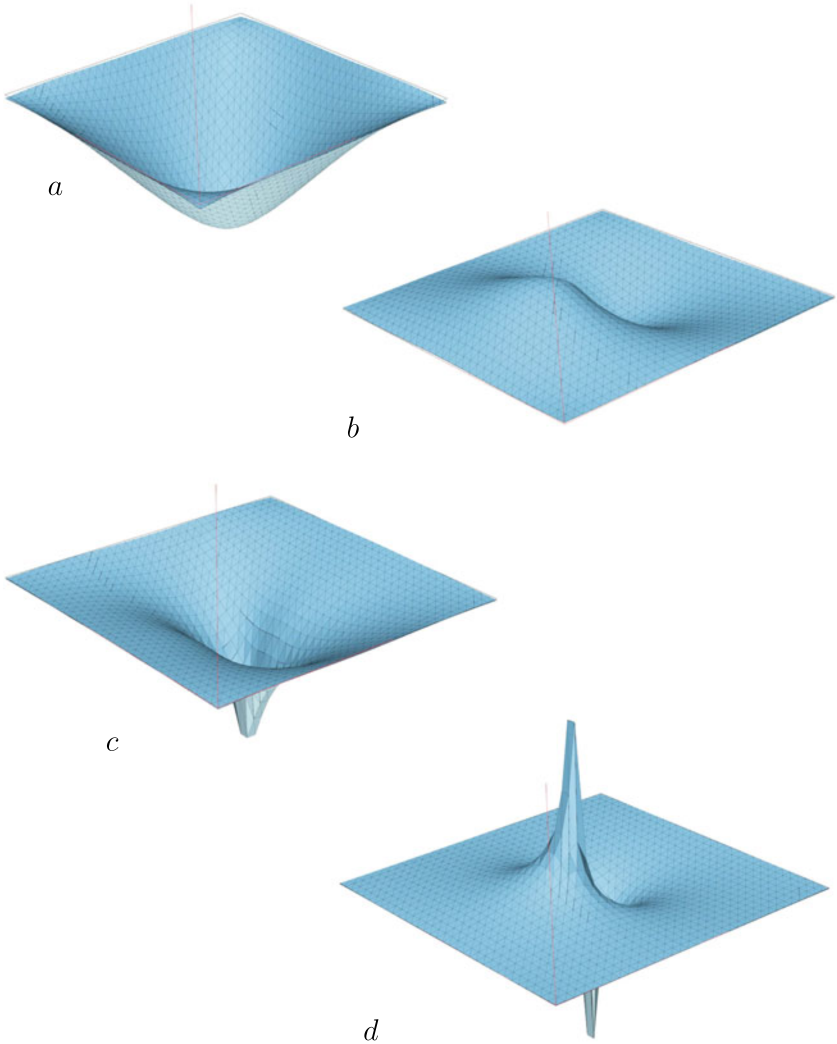


Fig. 2.42 Influence functions are generated by monopoles (on the left) or dipoles (on the right), influence function for **a** deflection, **b** rotation $w,_x$, **c** moment m_{xx}, **d** shear force q_x

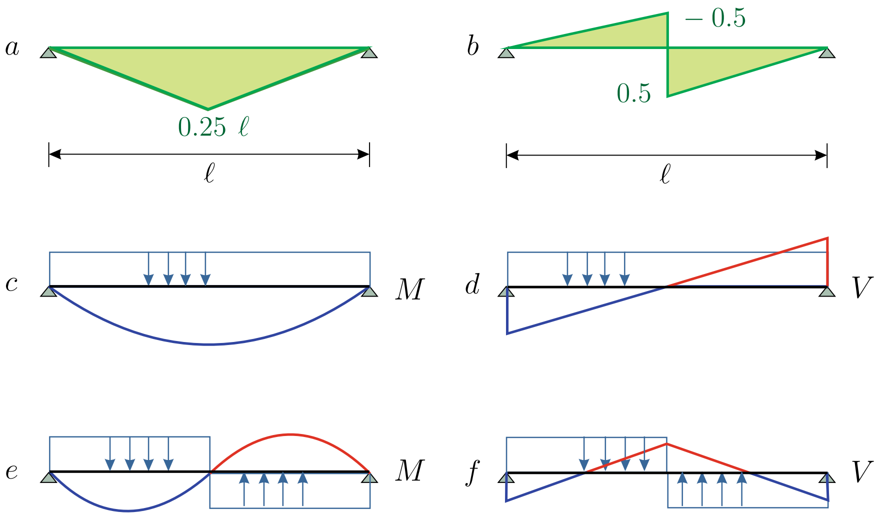


Fig. 2.43 Upper row influence functions for **a** the moment and **b** the shear force at the midpoint, **c** and **d** moments and shear forces under **e** symmetric and **f** antisymmetric load

motion is hindered by the foundation and so the other part must go all the way—alone—to effect a unit dislocation, $[[u]] = 1$, and so the influence function becomes a one-sided integral.

Remark 2.7 Not all influence functions tend to zero. If parts of the structure (after the installation of a hinge) can perform rigid body motions influence functions may blow up, see Fig. 2.46b.

Remark 2.8 The speed with which influence functions decay, depends on the order of the derivative of the target value, which would be 0, 1, 2, 3 in a beam

$$w(x), \quad w'(x), \quad M(x) = -EI\, w''(x), \quad V(x) = -EI\, w'''(x)\,. \tag{2.119}$$

The lower the order, the more evenly the influence function spreads out, and the more slowly it decays, as the influence function for the deflection $w(\boldsymbol{x})$ of a plate demonstrates, see Fig. 2.45 a. The influence function for the shear force q_x instead is a tightly packed dipole, see Fig. 2.45b, two infinitely large peaks, but the downswing from these peaks is equally steep and fast.

The particular behavior, of course, also depends on the support conditions, see Figs. 2.47 and 2.48, because structures with large overhanging parts (cantilevers) play a special role in this regard. Such parts can swing widely and they can easily blow up any influence function.

Influence functions for forces in statically determinate systems also deserve a remark. Since such systems are kinematic (after the installation of a hinge), deformations can develop **unhindered** because no energy is consumed. Nothing can prevent

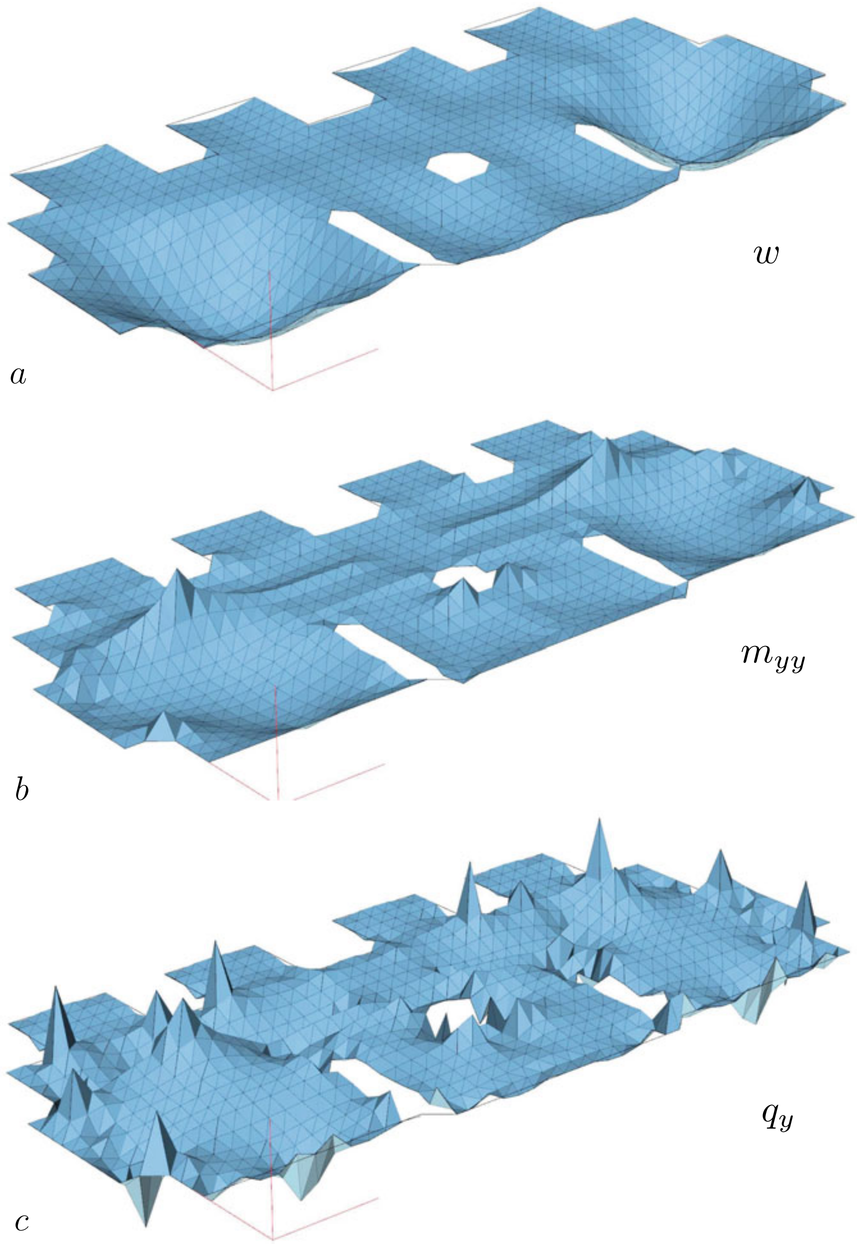


Fig. 2.44 Increasing complexity in a plate, **a** deflection w, **b** moments m_{yy}, **c** shear forces q_y

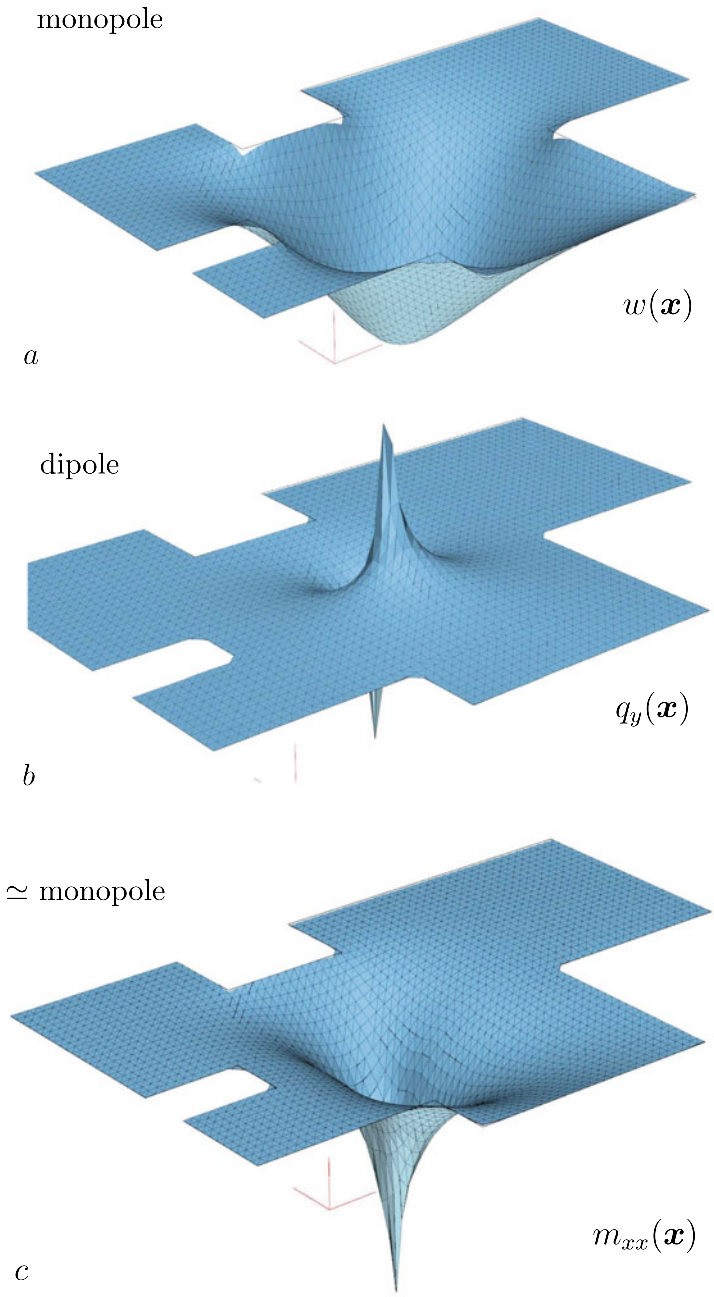


Fig. 2.45 Plate and influence functions, **a** for a deflection ($G_0 = O(r^2 \ln r)$), **b** a shear force ($G_3 = O(r^{-1})$), **c** a moment ($G_2 = O(\ln r)$), see (6.36) page 383 last column of the matrix

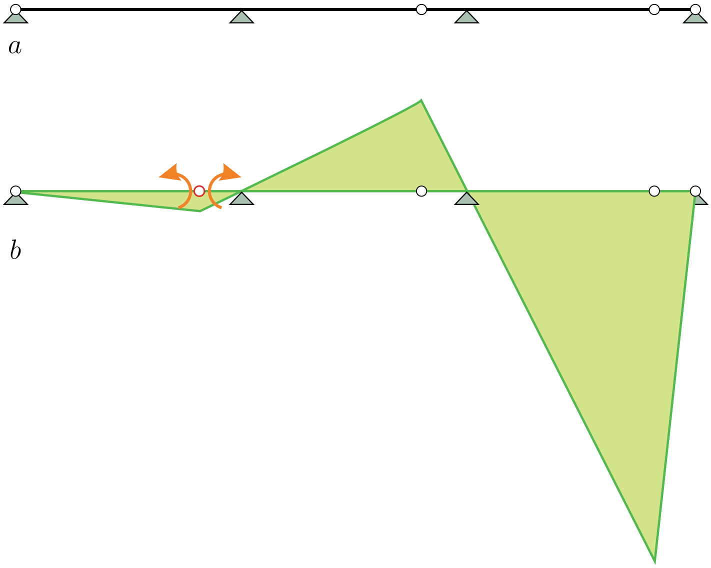


Fig. 2.46 **a** Gerber beam, **b** influence functions for a moment M; not all influence functions decay!

the influence function for the moment in a cantilever beam from running the part to the right of the source point under 45° into the sky, because it costs nothing. This is the reason why kinematic structures collapse so easily, since no energy is needed to trigger the collapse.

Statically indeterminate systems instead dampen the propagation of influence functions, while statically determinate systems lack such a barrier.

2.16 The Leaning Tower of Pisa

The symmetry and antisymmetry, which we observe in influence functions, also plays a prominent role in the leaning tower of Pisa.

The problem is that the soil stiffness under the tower is not uniform, and so the influence function $G_1(y, x)$ for the rotation of the foundation plate is not perfectly antisymmetric, but rather has a bias towards the softer side, and so the tower leans to this side, see Fig. 2.49,

$$w'(x_c) = \int_0^l G_1(y, x_c)\, p(y)\, dy\,. \tag{2.120}$$

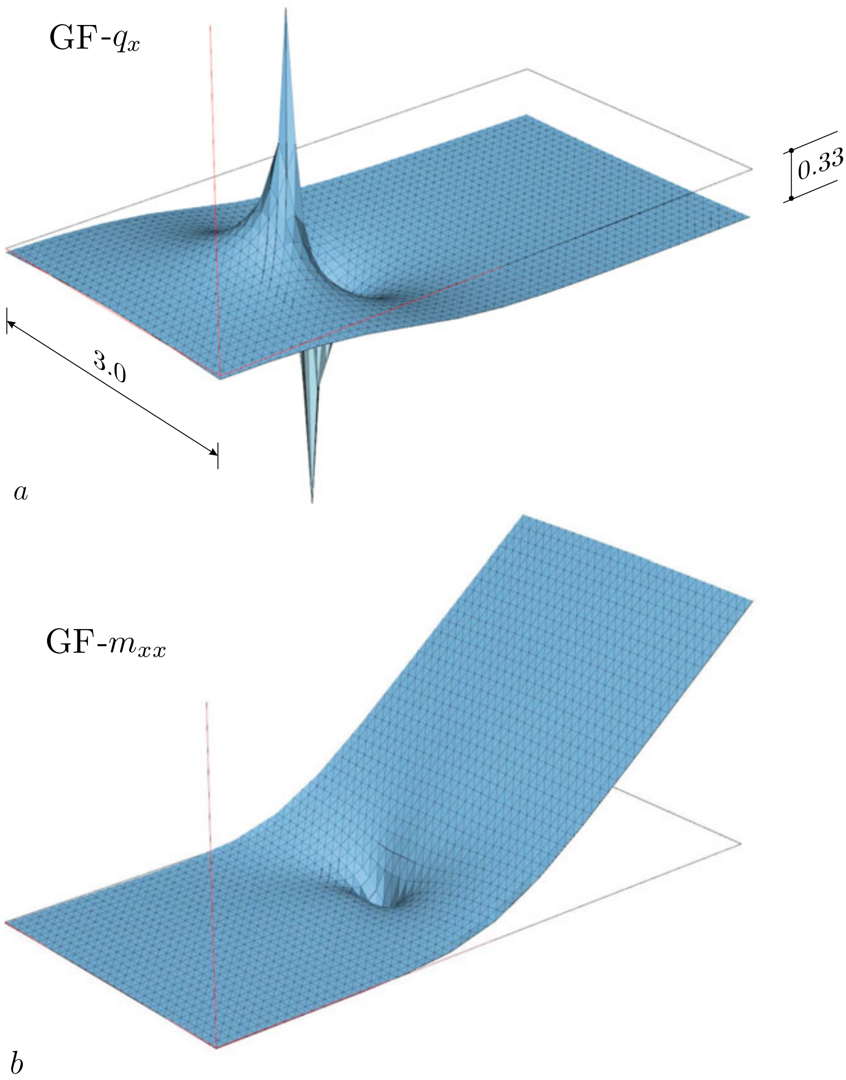


Fig. 2.47 Cantilever plate, **a** influence functions for a shear force q_x, and **b** the moment m_{xx}; it is amazing how with a "numerical" dislocation and a "numerical" kink it is possible to generate a nearly uniform dislocation and 45° rotations. How close can you come to the source point before the singularity, $O(1/r)$ in figure a and $O(\ln r)$ in figure b, dominates the scene?

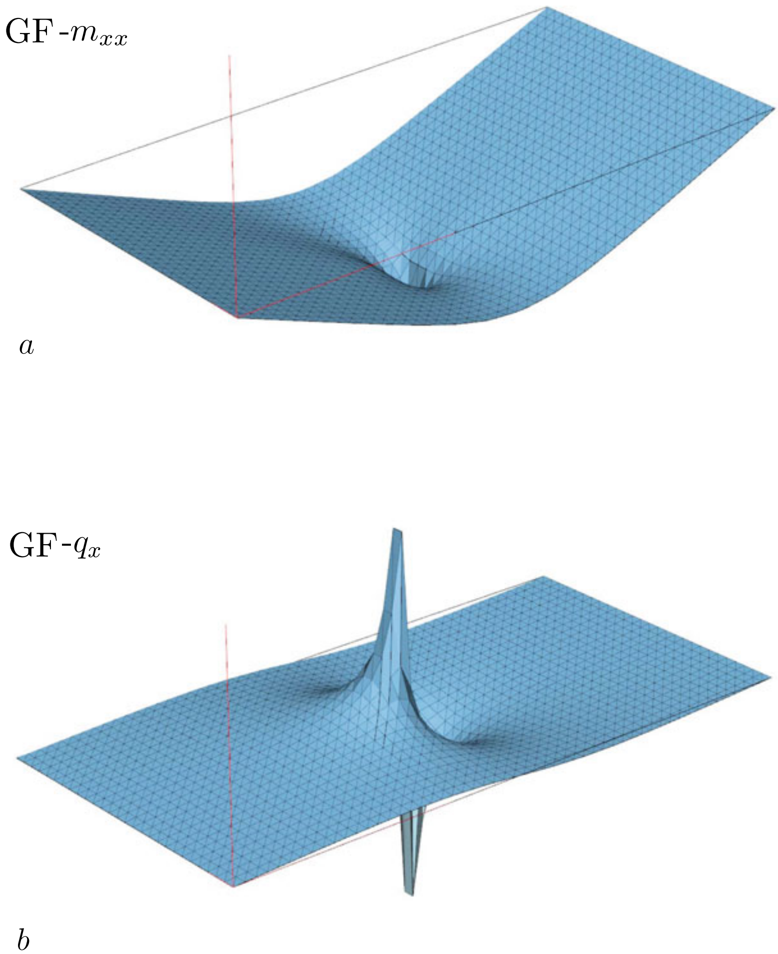


Fig. 2.48 Bridge, **a** influence function for the moment m_{xx}, and **b** for the shear force q_x at the midpoint; the influence function for the integral of q_x across the bridge should be identical with the beam solution

An antisymmetric kernel equals a balance, but if the balance is **misaligned**, even a perfectly symmetric load will rotate the balance. So, a zero rotation requires a perfectly antisymmetric kernel and a perfectly symmetric load.

Exercise: Given a non-perfect antisymmetric kernel, find the load p so that the rotation is zero.

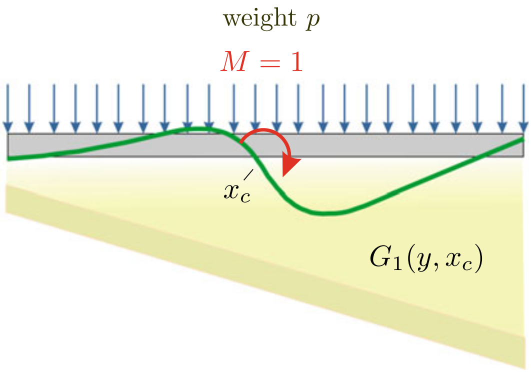


Fig. 2.49 The soil under the leaning tower of Pisa

2.17 Influence Functions for Integral Values

Oscillations near singular points are best smoothed out by **averaging**, by integrating over a short stretch.

Why this helps becomes clear when we look at the influence functions. The influence function for the stress σ_{yy} at a point is a spread of the source point in vertical direction, see Fig. 2.50b, but no mesh can simulate such a spread. If we "stretch" the point, turn it into a short line ℓ, and calculate with the mean of the stresses along this line

$$\sigma_{yy}^{\varnothing} = \frac{1}{\ell} \int_0^{\ell} \sigma_{yy}\, ds\,, \tag{2.121}$$

the influence function is a **simultaneous shift** of all points on the line in vertical direction, and such a displacement is easier to approximate with finite elements than a dislocation at one point. This is the reason why averaging gives better results.

If, as in Fig. 2.50a, the line is a complete cut through the plate, the integral of the stresses

$$N_y = \int_0^{l} \sigma_{yy}\, dx \tag{2.122}$$

is even exact, since this lift lies in $\mathcal{V}_h^+$ (= $\mathcal{V}_h$ + rigid body motions).

The beneficial effect of an integral measure on the bending moment in a plate is also evident in Fig. 2.51.

Influence functions of integral values are based on the same equation $\boldsymbol{K}\boldsymbol{g} = \boldsymbol{j}$ as before. If $J(w)$ is the mean value of the deflection in an element (x_a, x_b),

$$J(w) = \frac{1}{(x_b - x_a)} \int_{x_a}^{x_b} w(x)\, dx\,, \tag{2.123}$$

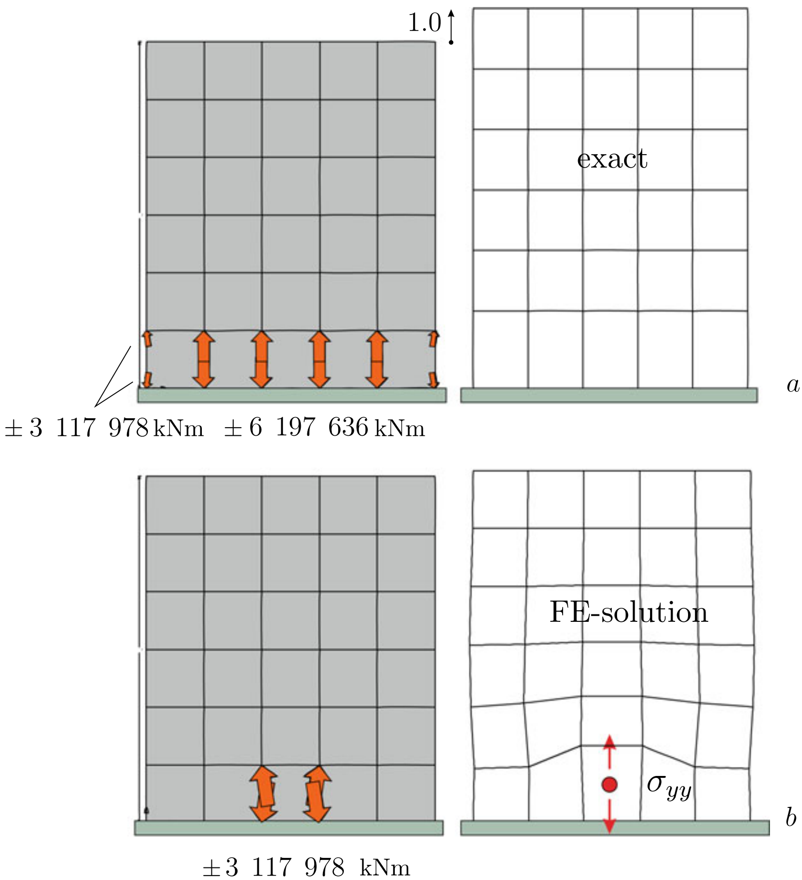


Fig. 2.50 Plate, **a** influence function for N_y (exact above the first row of elements), **b** influence function for σ_{yy} at a point

the j_i are the mean values of the φ_i

$$j_i = \frac{1}{(x_b - x_a)} \int_{x_a}^{x_b} \varphi_i(x)\, dx\,. \tag{2.124}$$

The influence functions for mean values of stresses are particularly simple to generate. The average stress $\sigma_{xx}^{\varnothing}$ in an element Ω_e is the integral

$$\sigma_{xx}^{\varnothing} = \frac{1}{|\Omega_e|} \int_{\Omega_e} \sigma_{xx}\, d\Omega = \frac{E}{|\Omega_e|} \int_{\Omega_e} (\varepsilon_{xx} + \nu\, \varepsilon_{yy})\, d\Omega\,, \tag{2.125}$$

but since $\varepsilon_{xx} = u_{x,x}$ and $\varepsilon_{yy} = u_{y,y}$, the domain integral can be replaced by a boundary integral over the edge Γ_e of the element

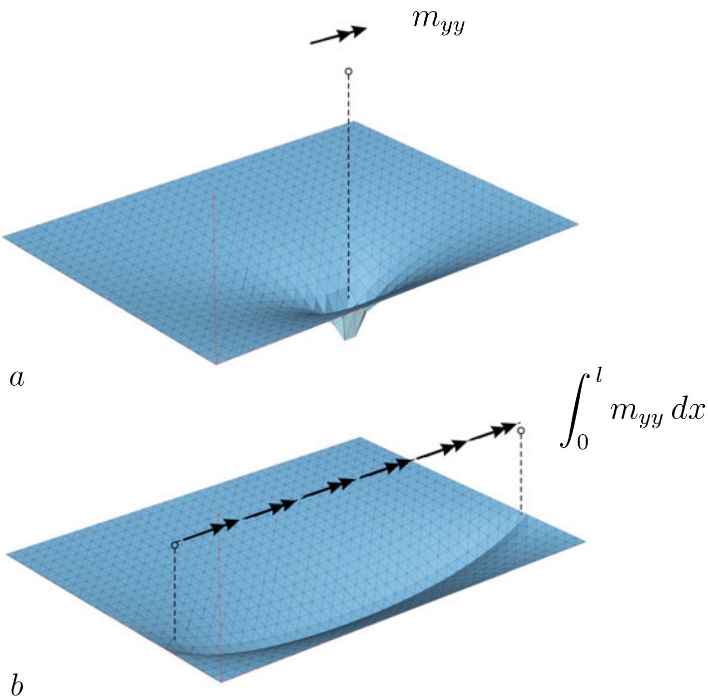


Fig. 2.51 Plate, **a** influence function for m_{yy}, and **b** for the integral of m_{yy}, [4]

$$\sigma_{xx}^{\varnothing} = \frac{E}{|\Omega_e|} \int_{\Omega_e} (\varepsilon_{xx} + \nu\, \varepsilon_{yy})\, d\Omega = \frac{E}{|\Omega_e|} \int_{\Gamma_e} (u_x\, n_x + \nu\, u_y\, n_y)\, ds\,, \tag{2.126}$$

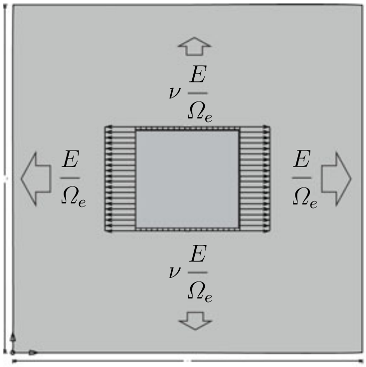
and because the influence function for the displacement u_x or u_y of a boundary point $\boldsymbol{x}$ is generated by a single force $P_x = 1$ or $P_y = 1$ placed at $\boldsymbol{x}$, the influence function for the boundary integral is the displacement field generated by horizontal and vertical line forces, $E/|\Omega_e| \cdot n_x$ and $E/|\Omega_e| \cdot n_y$, along the edge of the element Γ_e, see Fig. 2.52.

An equations says it directly: Let $\boldsymbol{G}_{\varnothing}$ the reaction of the plate to the edge forces $E/|\Omega_e| \cdot n_x$ and $\nu E/|\Omega_e| \cdot n_y$ in Fig. 2.52, which we write as vector $\boldsymbol{t}$, and let $\boldsymbol{u}$ the displacement field of the plate then

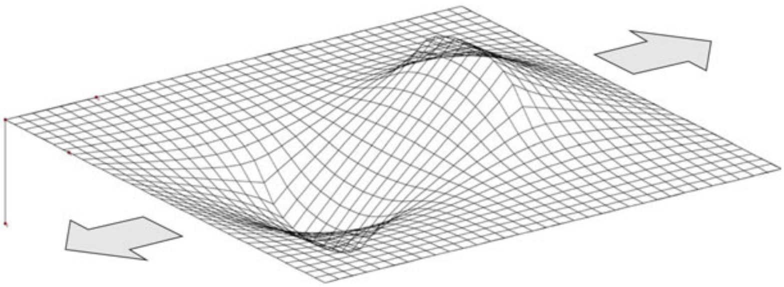
$$W_{1,2} = \int_{\Omega} \boldsymbol{G}_{\varnothing} \bullet \boldsymbol{p}\, d\Omega_{\boldsymbol{y}} = \int_{\Gamma} \boldsymbol{t} \bullet \boldsymbol{u}\, ds_{\boldsymbol{y}} = W_{2,1}\,, \tag{2.127}$$

and the second integral is identical with (2.126).

The **mean stresses** in a plate held fixed at its edge are therefore zero, since the edge forces which generate the influence function cannot displace the fixed edge.



a



b

Fig. 2.52 Influence function for the average value of σ_{xx} in an element, **a** the "Dirac delta" consists of horizontal forces along the vertical edge and (small, ν-fold) vertical forces along the horizontal edge of Ω_e, **b** the horizontal displacements of the Green's function, here plotted in z-direction

The same is true with slabs: The mean values of the moments of a slab, clamped on all sides, are zero. In the 1-D case we have encountered this phenomenon already in Chap. 1, Eqs. (1.31) and (1.32).

Rule of thumbs: When the edge is fixed the average values of the derivatives are zero.

The stresses in the middle of an element are the most accurate. On the one hand, because the midpoint is the point furthest away from the jumps at the edge, and on the other hand because a dislocation at the midpoint is "relatively" easy to generate. With bilinear elements it is even so, that the influence functions for the stresses at the center and the mean values of the stresses coincide [4]. The stresses at the midpoint of a bilinear element are mean values.

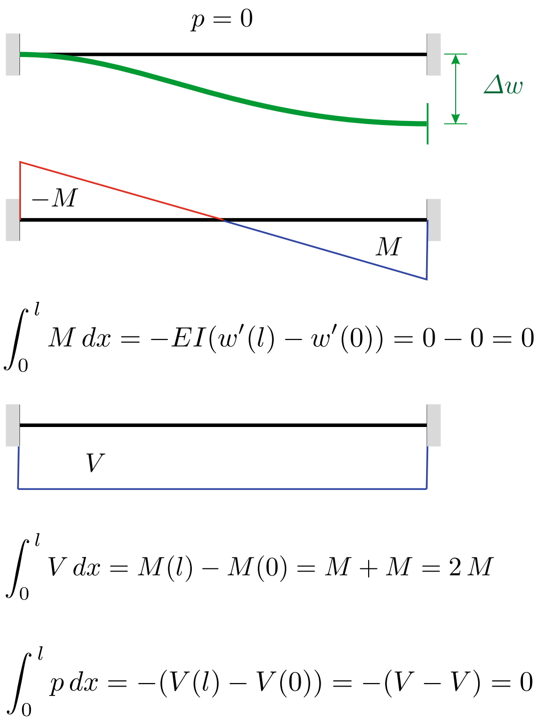


Fig. 2.53 Relations between integrals and end values

Integration by parts is also a statement about the **mean value** of the derivative of a function

$$\frac{1}{l}\int_0^l u'(x)\,dx = \frac{1}{l}(u(l) - u(0))\,. \tag{2.128}$$

This is why the integral of the normal force $N = EA\,u'(x)$ is proportional to the relative displacement of the beam ends

$$\int_0^l N(x)\,dx = EA\,(u(l) - u(0))\,, \tag{2.129}$$

and the integral of $M(x) = -EI\,w''(x)$ is a measure of the relative rotations of the end points, see Fig. 2.53,

$$\int_0^l M(x)\,dx = -EI\,(w'(l) - w'(0))\,. \tag{2.130}$$

The integral of the shear force

$$\int_0^l V(x)\,dx = M(l) - M(0) \tag{2.131}$$

equals the difference between the moments, and the distributed load $p = -V' = EI\,w^{IV}$ is responsible for the difference between the shear forces

$$\int_0^l p\,dx = -(V(l) - V(0))\,. \tag{2.132}$$

2.18 Influence Functions Integrate

Differentiating, $w \to w' \to w'' \to \ldots$ we move forward and integrating we move back. The influence function $G_1(y, x)$ for the normal force integrates the load once

$$N(x) = \int_0^l G_1(y, x)\,p(y)\,dy \qquad ('') \to (')\,, \tag{2.133}$$

and the influence function $G_0(y, x)$ for the displacement $u(x)$ integrates it twice

$$u(x) = \int_0^l G_0(y, x)\,p(y)\,dy \qquad ('') \to (\,)\,. \tag{2.134}$$

Fig. 2.54 illustrates the situation. The deflection w of the cantilever beam is the triple indefinite integral of the shear force $V = -EI\,w'''$

$$w = -\int\!\!\int\!\!\int V\,dx\,dx\,dx = -\int\!\!\int\!\!\int P\,dx\,dx\,dx\,, \tag{2.135}$$

and this explains the ℓ^3 in the end deflection

$$w(\ell) = \frac{P\,\ell^3}{3\,EI}\,. \tag{2.136}$$

When a moment $M = -EI\,w''$ is applied we see an ℓ^2

$$w(\ell) = \frac{M\,\ell^2}{2\,EI}\,, \tag{2.137}$$

while a distributed load p produces the deflection

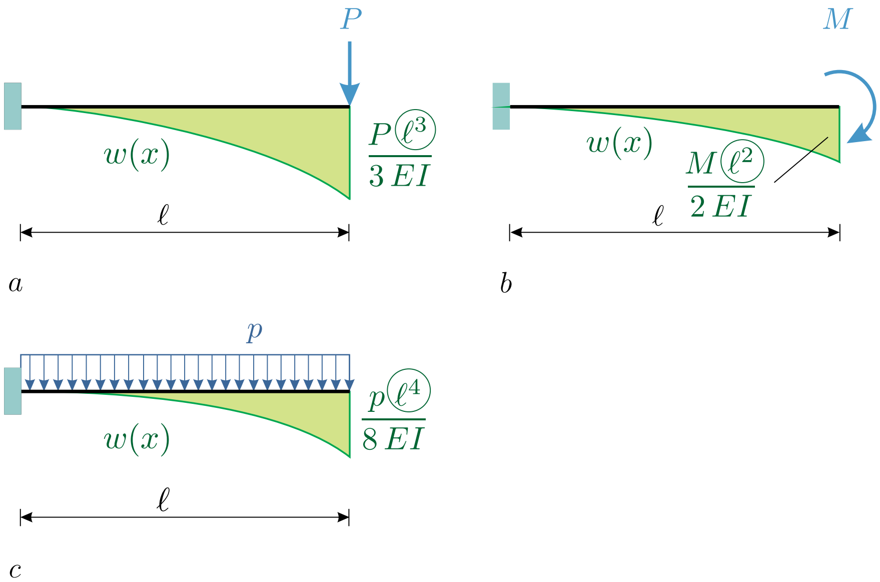
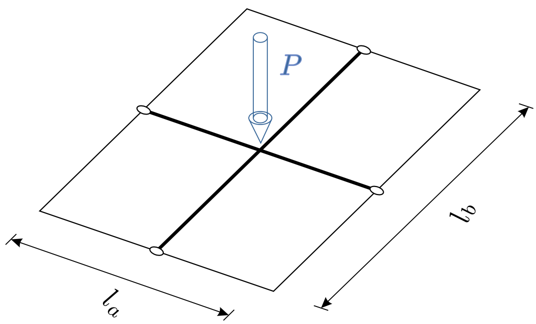


Fig. 2.54 Deflection of a cantilever beam due to **a** a single force—integrate three times, **b** a moment—integrate two times—and **c** distributed load—integrate four times

Fig. 2.55 Grid



$$w(\ell) = \frac{p\,\ell^4}{8\,EI}\,, \tag{2.138}$$

where the ℓ^4 matches the fourth-order derivative $EI\, w^{IV} = p$.

We find the l^3 of (2.136) also in the formula

$$\frac{P_a}{P_b} = \frac{l_b^3}{l_a^3}\,, \tag{2.139}$$

which regulates how a point load $P = P_a + P_b$ is split between two beams, see Fig. 2.55.

A stiffness matrix, $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$, differentiates, and therefore we find the "inverse" factor EI/l^3 and EA/l up front in the case of a beam and a bar element respectively.

On moving back it is important to know how we arrived at p. Differentiate the function $u(x) = \sin(\pi\, x/l)$ two times and four times respectively

$$-u'' = (\frac{\pi}{l})^2 \sin(\pi\, x/l) \qquad = p(x) \tag{2.140}$$

$$EI\, u^{IV} = (\frac{\pi}{l})^4 \sin(\pi\, x/l) \qquad = \bar{p}(x)\,. \tag{2.141}$$

In the first case it is the response of a rope to a load $p(x)$, and in the second case it is the deflection of a beam which carries a load $\bar{p}(x)$.

To calculate u at the point $x = l/2$ by integrating the right sides $p(x)$ and $\bar{p}(x)$ different influence functions are required, although we ask for the same value, $u(l/2) = \sin(0.5 \cdot \pi)$. We must know which operator created p. Where do the data come from?

2.19 St. Venant's Principle

According to this principle *"the difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from load"* [5].

The reason is simple: Effects spread via influence functions, which are scalar products of the load p and a kernel $G(y, x)$, and the kernel (usually) tends to zero with increasing distance from the source point. If the distance is large enough, a **one-point quadrature** is good enough and the load can be replaced by its resultant. But equivalent loads have the same resultant, and therefore an exchange will have no effect at a large enough distance.

The effects of antisymmetric loads, having zero resultants, in particular subside quickly. And if an antisymmetric load meets a "flat" influence function—no slope—the influence is zero from the start, *symmetric* × *antisymmetric = 0*.

Antisymmetric loads "differentiate" the influence functions.

This effect plays a significant role in the case of the forces f^+ in Chap. 5.

Fourier

A closely related topic are the effects of oscillating loads, since these have the tendency to cancel each other out, see Fig. 2.56. This is why Fourier series and the

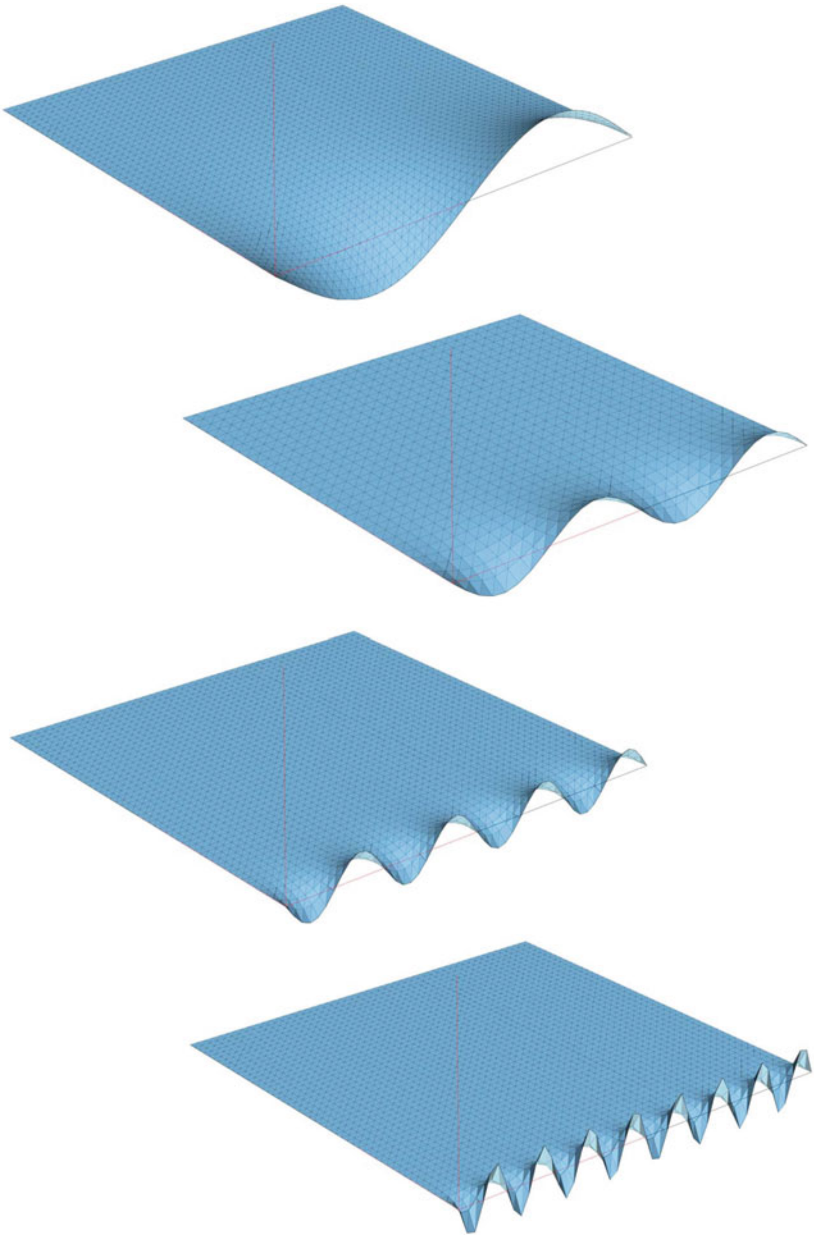


Fig. 2.56 Membrane with movable edge. The shorter the wave length of the induced edge deflection is, the more rapidly it decreases, an effect which is due to the influence functions. Rapidly oscillating line loads on a plate generate probably zero displacements at one meter distance

JPEG method[2] succeed. The higher frequencies in a Fourier series, the short wave oscillations do not get very far.

Decomposing a signal into its frequencies is one thing, but—no less important—is that effects are the **superposition of the signal with influence functions**.

The idea of the **discrete Fourier transform** is to sort a vector by its wave components and to omit the short waves, the high frequencies. The most important vector is the one-vector $\boldsymbol{u} = \{1, 1, 1, \ldots\}^T$, whose integral is simply the area of the influence function. This vector is followed by vectors, which resemble "frozen" waves of increasing frequency. Their effect (= integral) tends to zero with increasing frequency, because they are extinguished when superposed with the influence functions.

The real "trick" of the discrete Fourier transform is the clever basis of wave vectors, "broad brushstrokes", which is much more effective than the "pointillist-like" basis of Euclidean unit vectors $\boldsymbol{e}_i$. If you want to transmit a vector $\boldsymbol{u} = \sum_{i=1}^{64} u_i \, \boldsymbol{e}_i$, you have to send all 64 components u_i, you may not omit one u_i, (it could just be the brightest pixel). Fourier instead can limit himself to the first 10 components without it being too obvious at the other end of the line that something is missing.

2.20 Second-Order Theory

Second-order beam theory is based on

$$EI\, w^{IV}(x) + P\, w''(x) = p(x) \tag{2.142}$$

where P is the compressive force in the beam and $p(x)$ is the distributed load, see Fig. 2.57a. This is a nice fourth-order linear self-adjoint differential equation with constant coefficients, but the problem lies in the coefficient P, which is load case dependent as can be seen in Fig. 2.57. Each P requires a different influence function for $w(x)$ and the section forces.

The more P nears the **buckling load** P_{crit}, the more the influence function for the rotation of the beam's end, Fig. 2.57 b, or for the deflection, Fig. 2.57c, bulge out.

In principle, second-order theory is a **nonlinear problem**, where the longitudinal displacement $u(x)$ and the lateral displacement $w(x)$ are linked in the system

$$-EA\left(u' + \frac{1}{2}\,(w')^2\right)' = p_x \tag{2.143a}$$

$$EI\, w^{IV} - \left(EA(u' + \frac{1}{2}\,(w')^2)\, w'\right)' = p_z\,. \tag{2.143b}$$

[2]In the JPEG method (DCT) a black and white image is divided into blocks of 8×8 pixels and the grayscales in the 64 pixels of a block form a vector, to which we apply a Fourier transform and we leave out the high frequencies.